MATH 135 Lecture Notes Sections 3.6 & 3.7 — Derivatives of Products, Quotients, and Powers of Functions

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In the previous lecture we learned how to find the derivative of the sum of two functions:

$$\frac{d}{dx}(u(x)+v(x)) = \frac{d(u+v)}{dx} = \frac{du}{dx}+\frac{dv}{dx}$$

In other words, the derivative of a sum is the sum of the derivatives. You might think that the same rule would apply to products, but it does not. You can show that in general the derivative of a product is not the product of the derivatives by considering two power functions.

Let $u = x^5$ and $v = x^3$ and let $y = uv = x^5 * x^3 = x^8$. Taking the derivative of each function we have

$\frac{du}{dx} = 5x^4$ and $\frac{dv}{dx} = 3x^2$. $\frac{du}{dx}\frac{dv}{dx} = 15x^6$ but $\frac{d(uv)}{dx} = 8x^7$. Products don't work like sums!

However, there *is* a way to find the derivative of the product of two functions. It appears as equation (3-12) on page 96. Learn this important rule.

Derivative of a Product:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

A derivation of this rule appears at the top of page 96. Let $u = x^5$ and $v = x^3$ and show that the product rule does give you $\frac{d(uv)}{dx} = 8x^7$.

Examples 2 & 3 at the bottom of page 96 show how the rule is applied.

There is also a rule for finding the derivative of a quotient of two functions.

Derivative of a Quotient (page 97):

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

(Think: square of the denominator goes in the denominator, then, in the numerator write denominator times derivative of numerator *minus* numerator times derivative of denominator.) A derivation of this rule appears at the top of page 97. Examples 4-6 on pages 97-98 show how the rule is applied.

Now we come to one of the most useful rules for finding derivatives. It applies when y function of u and u is a function of x.

The **Chain Rule** (page 100)

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

(Remember this rule by thinking of the *du*'s in numerator and denominator canceling each other out.)

Here is how the chain rule may be applied in Example 1, page 100. Compare this approach with the book's approach.

$y = (3 - 2x)^3$. Think of the expression in parentheses as u. Then $u = 3 - 2x$ and $y = u^3$. Take the derivative of each function: $\frac{du}{dx} = -2$ and $\frac{dy}{du} = 3u^2$

Therefore $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(-2) = -6u^2$. However, we are not finished since we have to substitute $u = 3 - 2x$ into this expression to obtain: $\frac{dy}{dx} = -6(3 - 2x)^2$

[Check by expanding $y = (3 - 2x)^3$ and taking the derivative. Expand $-6(3 - 2x)^2$ to get the same result.]

Study Examples 2 and 3 on page 101.

Recall the definitions of negative and rational exponents (page 8, equations 1-9 and 1-10):

$a^{-n} = \frac{1}{a^n}$ where n is a positive integer, and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ where m and n are both integers.

Our rule for finding derivatives of powers may now be extended to all rational numbers, positive and negative. See equation 3-16 on page 101.

To use this **Expanded Power Rule,** it helps to rewrite functions in exponential form.

Example A. Find the derivative of $y = \frac{1}{x^4}$.

Solution. Rewrite $y = x^{-4}$. Then $\frac{dy}{dx} = -4x^{-5} = \frac{-4}{x^5}$ (Note $^-4 - 1 = {}^-5$.)

Example B. Find the derivative of $y = \sqrt{x}$

Solution. Rewrite $y = x^{\frac{1}{2}}$. Then $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$. (Note $\frac{1}{2} - 1 = -\frac{1}{2}$)

Study examples 4-8, pages 102-104, for applications of the expanded power rule in conjunction with the product, quotient, and chain rules.

Exercises: pages 98-99: 1, 3, 7, 9, 13, 19, 33, 35, 37, 39
pages: 104-105; 3, 5, 7, 9, 11, 13, 17, 21, 29, 31, 35